AP Physics C: Mechanics Review

Types of Error (in labs)

1. **System Error**

Materials, Air Resistance, Friction

2. **Mathematical Error**

Truncating, Calculation error

3. **Observational Error**

Parallax, Reaction time, measuring distance

UNIT 1: 1-D Motion (with Calculus)

Fundamentals of Unit 1 Physics

Derivative : slope of a tangent line at a point on a function

- Derivative of f(x) is denoted as $\frac{df(x)}{dx}$ or f'(x)
- 2nd derivative of f(x) is denoted as $\frac{d}{dx}(\frac{df(x)}{dx})$ or f''(x)
- Common Derivatives: (A and B are constants)
    - $f(x) = A$ $\quad f'(x) = 0$
    - $f(x) = Ax + 0$ $\quad f'(x) = A$
    - $f(x) = Ax^n$ $\quad f'(x) = Anx^{n-1}$ $\quad f''(x) = An(n-1)x^{n-2}$
    - $f(x) = A\sin(Bx)$ $\quad f'(x) = AB\cos(Bx)$ $\quad f''(x) = -AB^2\sin(Bx)$
    - $f(x) = A\cos(Bx)$ $\quad f'(x) = -AB\sin(Bx)$ $\quad f''(x) = -AB^2\cos(Bx)$
    - $f(x) = Ae^{Bx}$ $\quad f'(x) = ABe^{Bx}$
    - $f(x) = \ln x$ $\quad f'(x) = \frac{1}{x}$

Kinematics

Instantaneous Velocity (m/s) : velocity of an object at that instant in time (slope of a position vs time graph at that time)

- $v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx(t)}{dt}$

Instantaneous Acceleration (m/s^2) : acceleration of an object at that instant in time (slope of a velocity vs time graph at that time)

- $a(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2}$

Jerk etc… : term for polynomials of max degree of 3 or more which means constant is not constant

Boundary Conditions : the conditions that define the boundary of a system or scenario

- Initial conditions are an example of boundary conditions

Deriving Kinematics with Calculus

$x(t) = C_1t^2 + C_2t + C_3 = ½ at^2 + V_o + x_o$

$v(t) = 2C_1t + C_2$

$a(t) = 2C_1$

$x(0) = x_o$ $\qquad v(0) = v_o$ $\qquad a(0) = a$

$x_o = C_1(0)^2 + C_2(0) + C_3$

$x_o = C_3$

$v_o = 2C_1(0) + C_2$

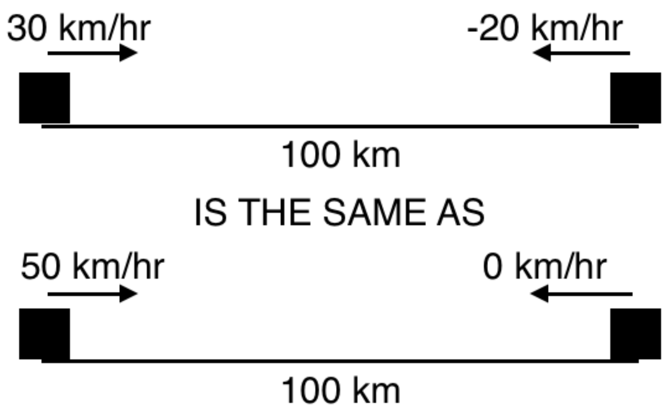
$v_o = C_2$

$a = 2C_1$

Relative Velocity

An object's velocity relative to another object

- Helps solve questions that initially seem difficult



TIPS ON HOW TO SOLVE PROBLEMS:

1. Derivatives for trig functions
    a. sin → cos → -sin → -cos → -(-sin) or sin → repeats
2. Use relative velocities when possible to solve problems involving two objects at different speeds

UNIT 2: Multidimensional Motion and Vectors

Fundamentals of Unit 2 Physics

Unit Vectors : vectors that have a magnitude of 1 and points along the +x-axis, +y-axis, or +z-axis

- |a| means magnitude of a
- For labeling vectors in 3D space (doesn't have to be 3D)
- $\hat{I}$ = "I hat" = x-axis = x component
- $\hat{j}$ = "J hat" = y-axis = y component
- $\hat{k}$ = "K hat" = z-axis = z component
- $r = r_x i + r_y j + r_z k$
- magnitude = $\sqrt{r_x^2 + r_y^2 + r_z^2}$

Right Handed Coordinate System/Rule : mnemonic for understanding orientation of vectors and axes in 3 dimensional space, used for cross products

- For a cross product, "Sweep" the first vector into the second vector, the thumb points toward direction of the third vector
- A x B = "Sweep" vector A into vector B, thumb is the direction of C

Taking derivatives of unit vectors

r = position vector
v = velocity vector
a = acceleration vector

$$v = \frac{dr}{dt} = \frac{dr_x i}{dt} + \frac{dr_y j}{dt} + \frac{dr_z k}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = v_x i + v_y j + v_z k$$

$$a = \frac{dv}{dt} = \frac{dv_x i}{dt} + \frac{dv_y j}{dt} + \frac{dv_z k}{dt} = \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{d^2z}{dt^2} = a_x i + a_y j + a_z k$$

Adding and subtracting vectors

For unit vectors:

Add or subtract i components, j components, and k components of all vectors to get the resultant vector

$\Delta r = r_f - r_o$

Vector Addition Rules:

A, B, and C are vectors p and q are constants

1. A + B = B + A
2. A + (B + C) = (A + B) + C
3. A + 0 = A [Null Vector Property]
4. A + (-A) = 0 [0 = null vector]
5. p(A + B) = (pA + pB) [Multiplicative property]

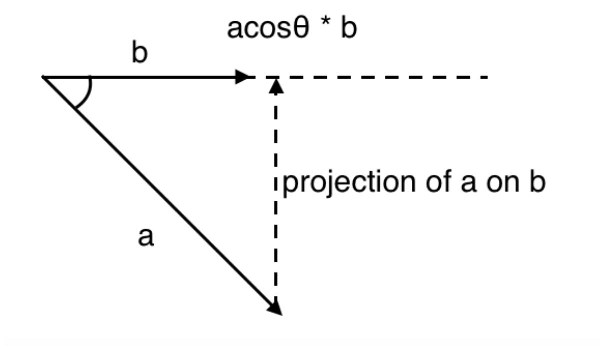
6. $(q + p)A = qA + pA$

Vector Products (dot and cross product)

Dot product → Scalar
Cross product → Perpendicular vector
$a \bullet b \neq a \times b$
Dot Product:
$i * j$ or $j * k$ or $k * i = 0$ ($1 * 1 * \cos 90^\circ = 0$ as these unit vectors are perpendicular to each other)
$i * i$ or $j * j$ or $k * k = 1$ ($1 * 1 * \cos 0^\circ = 1$)
Cross Product:
$i * j = k$
$j * k = i$
$k * i = j$

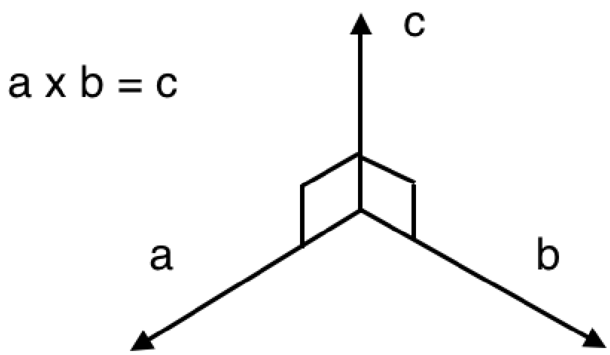
Dot Product : dot product of 2 vectors occur when the 2 vectors are "dotted" and results in a **scalar** quantity

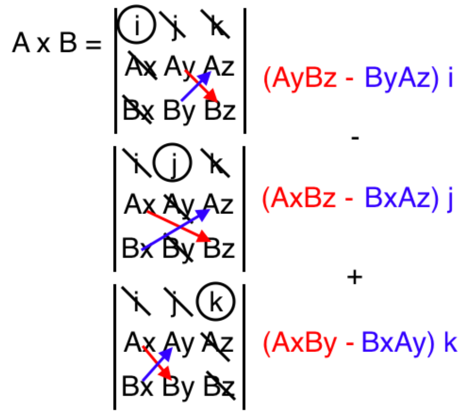
- $|a \bullet b|$ = a projection of vector a along vector b then scaled by magnitude of b
- $|a \bullet b| = |a||b|\cos\theta$
    - θ = angle between vector a and b
    - This formula is useful for obtaining the angle between the two vectors
        - Find θ with $\cos^{-1}(|a \bullet b| / |a||b|)$
- $a \bullet b = (a_x i * b_x i) + (a_y j * b_y j) + (a_z k * b_z k)$ = scalar quantity
- Example: Work is a dot product of Force and Displacement
    - $W = F \bullet \Delta x$



Cross Product : the cross product of two vectors occurs when two vectors are "crossed" [multiplied] in such a way that it results in another vector

- $|a \times b|$ = vector c that is orthogonal (perpendicular) to both vector a and b
- $|a \times b| = |a||b|\sin\theta$
- Find direction of 3rd vector (c) or resulting vector by using right hand rule
- $a \times b = (A_yB_z - B_yA_z)i - (A_xB_z - B_xA_z)j + (A_xB_y - B_xA_y)k$
- Example: Torque is the cross product of Radius and Force
    - $\tau = r \times F$





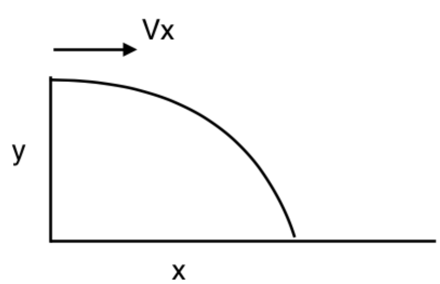
3D Kinematics

Same as 2D kinematics, but have 3 columns instead for x-direction, y-direction, and z-direction.
Apply kinematic equations if acceleration is present and $\Delta x = V * t$ if acceleration isn't present
Answer in unit vector form (i, j, and k)
Velocity and acceleration functions can be derived from the position function using derivatives
MAKE SURE UNITS ARE PRESENT

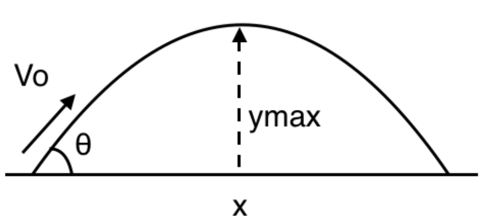
Applications of Multidimensional Motion

Simple 2D Kinematics

Projectiles:



$V_x(\Delta x, \Delta y) = \Delta x\sqrt{\frac{-5}{\Delta y}}$



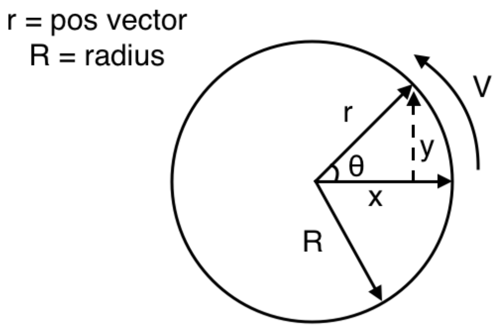
Time independent equations:

$\Delta x = \frac{2V_o^2 sin\theta cos\theta}{g}$ $\Delta y = \frac{V_o^2 sin^2\theta}{2g}$ $t = \frac{2V_o sin\theta}{g}$

Equation of the path (trajectory):

$y(x) = x\tan\theta - \frac{gx^2}{2V_o^2 cos^2\theta}$

Centripetal Motion combined with unit vectors and calculus



$\theta = \omega * t$

$r = xi + yj$

$r = (r\cos\theta)i + (r\sin\theta)j$

$|r| = \sqrt{R^2 cos^2\theta + R^2 sin^2\theta}$

$|r| = \sqrt{R^2 (cos^2\theta + sin^2\theta)}$

$|r| = \sqrt{R^2 (1)}$

$|r| = R$

Substitute $\theta = \omega * t$ into vector r

$r = (r\cos(\omega t))i + (r\sin(\omega t))j$

$v = (-R\omega\sin(\omega t))i + (R\omega\cos(\omega t))j$

$a = (-R\omega^2\cos(\omega t))i - (R\omega^2\sin(\omega t))j$

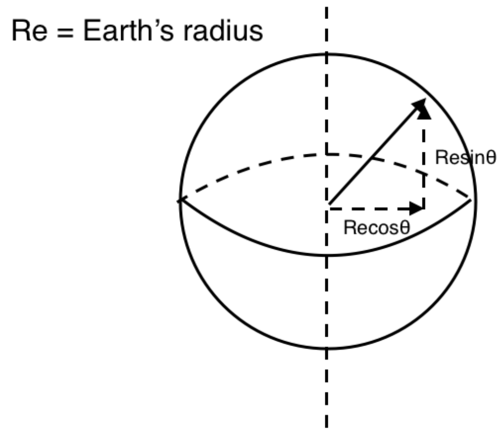
$a = -\omega^2[(R\cos(\omega t)i + (R\sin(\omega t))j]$ [negative of vector r is vector a, opposite direction (towards center)]

$|a| = -\omega^2$(r vector)

$|a| = -\omega^2 R$

$|a| = (\frac{v^2}{R^2})R$

$|a| = v^2 / R$



@ equator: ($r = r_E$ because there's no y-component, position vector is the radius of the Earth)

$R_E = r_e$

$a_c = v^2 / R_E$

$a_c = (r\omega)^2 / r_E$

$a_c = r\omega^2$

$a_c = R_E\omega^2$

$a_c = 6.38 \times 10^6 \left[\frac{2\pi}{24 * 3600}\right]^2$

$a_c = 0.03$ m/s^2

TIPS ON HOW TO SOLVE PROBLEMS:

1. For cross products, draw out the matrix/table to easily get the resulting vector
2. The projection of vector a on vector b is |a|cosθ. The projected vector will have the cosθ next to it.

UNIT 3: Forces and Newton's Laws of Motion

Fundamentals of Unit 3 Physics

Forces - 4 Fundamental Forces (Field Forces)

1. Gravitational Force
2. Electromagnetic Force
3. Weak Nuclear Force (2nd strongest)
4. Strong Nuclear Force (strongest)

Field Forces : there is no physical contact, but the field interacts with the object

- Not Tension, Drag, Friction, or Normal Force

Contact Forces : here is physical contact between objects

Newton's Laws of Motion

NOT VALID in an accelerating or rotating reference frame

Newton's 1st Law (N1L) : In an inertial reference frame, an object at rest stays at rest and an object in motion continues its motion along a straight line path with constant velocity unless acted upon by a non-zero net force (unbalanced force)

- $\sum F = 0$
- Examples/Problems:
  - Tension in a cable
  - Drag Force and Terminal Velocity

Newton's 2nd Law (N2L) : In an inertial reference frame, a non zero net external force causes an object to accelerate. The acceleration of the object is inversely proportional to the mass of the object and directly proportional to the magnitude of the net external force,

- $\sum F = ma = m\frac{dv}{dt} = \frac{d^2r}{dt^2}$
- $\sum F = \frac{dp}{dt}$ (Net external force = derivative of momentum)
- Examples/Problems:
  - Elevator Problems
  - Mass-Pulley System
  - Incline Plane
    - $a = g\sin\theta$ (frictionless)
    - $F_N = mg\cos\theta$
  - Pulley and Incline Planes
  - Bank Curves
    - $F_N = mg / \cos\theta$
      - Different from Incline Plane because follows centripetal path and F_N needs to be greater to supply the centripetal acceleration (F_{NX})
  - Rollercoaster Problem

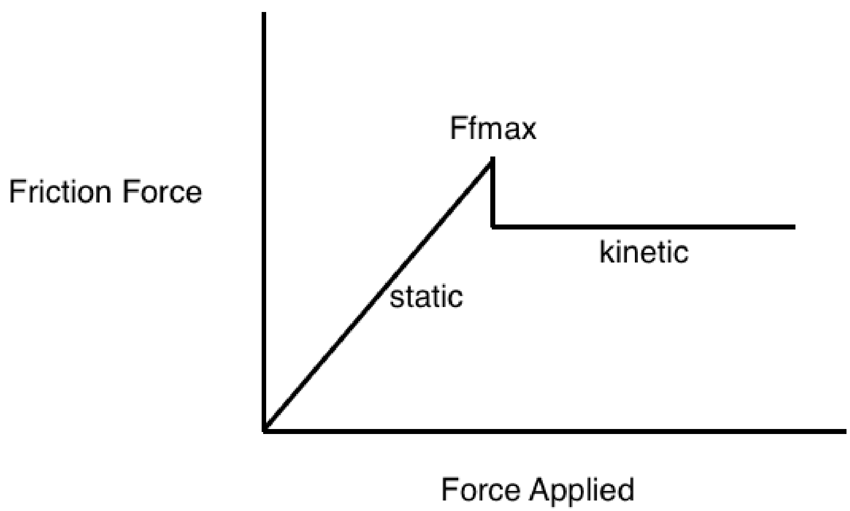
Forces of Friction

Force of Static Friction (F_{fs})

- $F_{fs} \leq \mu_s F_N$ (Inequality)
- $F_{fmax} = \mu_s F_N$ (maximum static friction force)
- Proportional to applied force until F_{fmax}, not constant
- Doesn't always oppose or prevent motion, sometimes it can start motion
- Can accelerate an object (walking)
- μ_s is a function of 2 surface textures in physical contact

Force of Kinetic Friction (F_{fk})

- $F_{fk} = \mu_k F_N$
- Always opposes motion of an object
- Approximately constant between two surfaces, no matter how much applied force
- Can accelerate an object (block slipping off another moving surface)
- μ_k is a function of 2 surface textures in physical contact



Critical Slip and Critical Kinetic Angle

Critical Slip Angle (θ_{cs}): Maximum angle for which the block doesn't slide down

$a = 0$

$a = g\sin\theta - \mu_s g\cos\theta$

$0 = g\sin\theta - \mu_s g\cos\theta$

$\mu_s g\cos\theta = g\sin\theta$

$\mu_s \cos\theta = \sin\theta$

$\mu_s = \tan\theta$

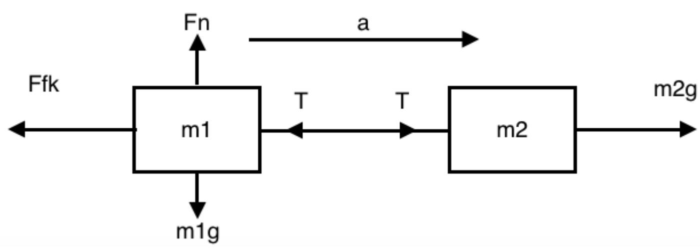
$\theta_{cs} = \tan^{-1}(\mu_s)$

Critical Kinetic Angle (θ_{ck}): Angle for which the block slides down with constant velocity

$a = 0$
$a = g\sin\theta - \mu_k g\cos\theta$
$0 = g\sin\theta - \mu_k g\cos\theta$
$\mu_k g\cos\theta = g\sin\theta$
$\mu_k \cos\theta = \sin\theta$
$\mu_k = \tan\theta$
$\theta_{ck,} = \tan^{-1}(\mu_k)$

Tug of War Diagram (TOW)

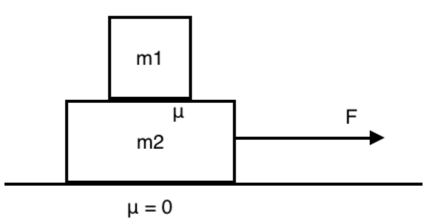
Internalizing forces between objects (Tension/Normal Force)
Useful for quickly calculating the acceleration of a system

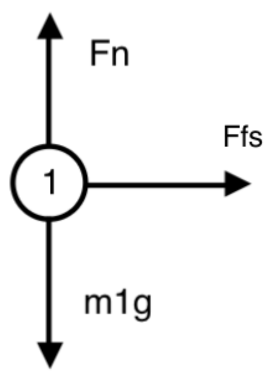


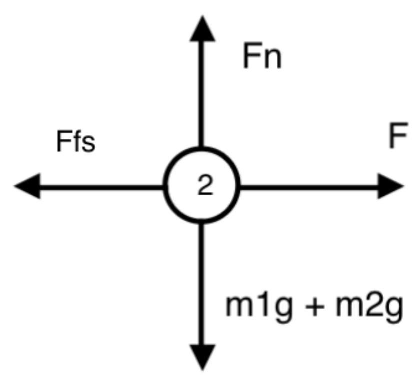
$\sum F = ma$
$m_2g - F_{fk} = (m_1 + m_2)a$
$m_2g - \mu_k F_N = (m_1 + m_2)a$
$m_2g - \mu_k m_1 g = (m_1 + m_2)a$
$a = \frac{g(m_2 - \mu_k m_1)}{m_1 + m_2}$

Tension can be solved by FBD with mass 1 or mass 2

Stacked Blocks

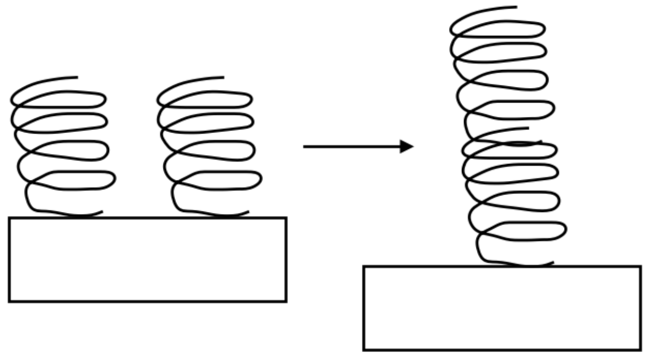






- Both masses are sliding on a frictionless surface
- Force keeping Mass 1 on top of Mass 2 is static friction
- If Mass 1 were to “slip off” of Mass 2, kinetic friction would provide the acceleration

Combining and Splitting Springs

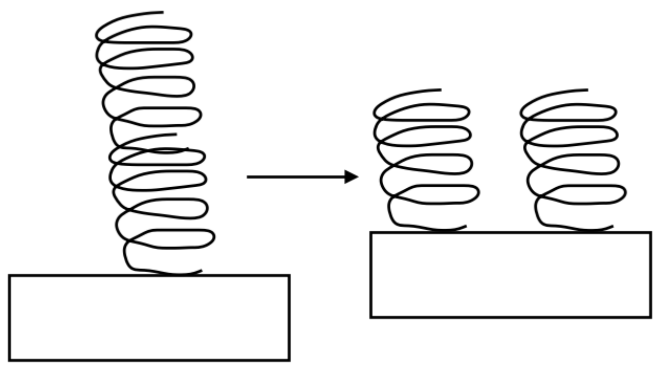


Converting parallel springs to series:
(Δy is the same for all springs)
(F is the sum for all springs)

$F_s = F_1 + F_2$

$K_{eq}\Delta y = K_1\Delta y + K_2\Delta y$

$K_{eq} = K_1 + K_2$



Converting series springs to parallel:
(F_s is the same for all springs)
(Δy is the sum for all springs)

$\Delta y_{eq} = \Delta y_1 + \Delta y_2$

$\frac{F}{K_{eq}} = \frac{F}{K_1} + \frac{F}{K_2}$

$\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2}$

$K_{eq} = \frac{1}{\frac{1}{K_1}+\frac{1}{K_2}}$

Centripetal Acceleration

$a_c = v^2 / r$

TIPS ON HOW TO SOLVE PROBLEMS:

1. Static and Kinetic Friction can both provide acceleration

UNIT 4: Work, Power, and Energy

Fundamentals of Unit 4 Physics

Work (J) : transfer of energy to, or from a system by the action of force

- $W = F\Delta x\cos\theta = F * \Delta x = \int F(x)\,dx$

- Work positive if F and Δx in same direction, KE increases
- Work negative if F and Δx in opposite direction, KE decreases
- Work is area under curve of F vs Δx graph (integral derived from chain rule)
- Work of something in orbit is 0 because F is perpendicular to Δx
- Dot product of F and Δx vectors

Power (W) : Rate of doing work, or rate of energy dissipation or rate of energy generation.
- Scalar quantity
- P = $\frac{dW}{dt}$ = F * v
- 1 watt of power is when 1 joule of work is performed in 1 second

Work - Kinetic Energy Theorem
W = ΔKE = KE_f - KE_o

Conservative vs Non-Conservative Forces
Conservative Force : the work done by the conservative forces is independent of the path and only depends on displacement
- Gravitational force, spring force, tension, normal, electrostatic

Non-Conservative Force : the work done by non-conservative forces dependent of the path taken
- Friction force, drag force, resistive forces

Potential Energy
Potential Energy (J) : Energy of a system due to its position relative to some reference point
- Potential energy of a system is the minimum work done by an external agent against a conservative force (external force like applied force)
  - If work done by external agent is positive, then system gained potential energy
    - **W_{ext} positive** means **ΔU increase**
  - If work done by external agent is positive, then system gained potential energy
    - **W_{ext} negative** means **ΔU decrease**
  - **W = ΔU for external agents/forces (applied force)**
- Potential energy of a system is the negative of the work done by the conservative force (internal force like gravity/spring)
  - If work done by internal agent is positive, then system lost potential energy
    - **W_{int} positive** means **ΔU decrease**
  - If work done by external agent is positive, then system gained potential energy
    - **W_{int} negative** means **ΔU increase**
  - **W = -ΔU for internal agents/forces (gravitational/spring force)**
  - **-ΔU = W =** $\int_A^B F(r)\, dr$
  - **ΔU = -W =** $-\int_A^B F(r)\, dr$ F(r) is conservative internal force (gravity/spring)
  - **F(r) =** $-\frac{dU}{dt}$ F(r) is conservative internal force (gravity/spring)

Orbit is bound if U_g is negative = required energy to break mass out of orbit
Orbit is unbound if U_g is positive = ~~required energy to place energy in orbit~~ (unsure)

Mechanical energy and its conservation

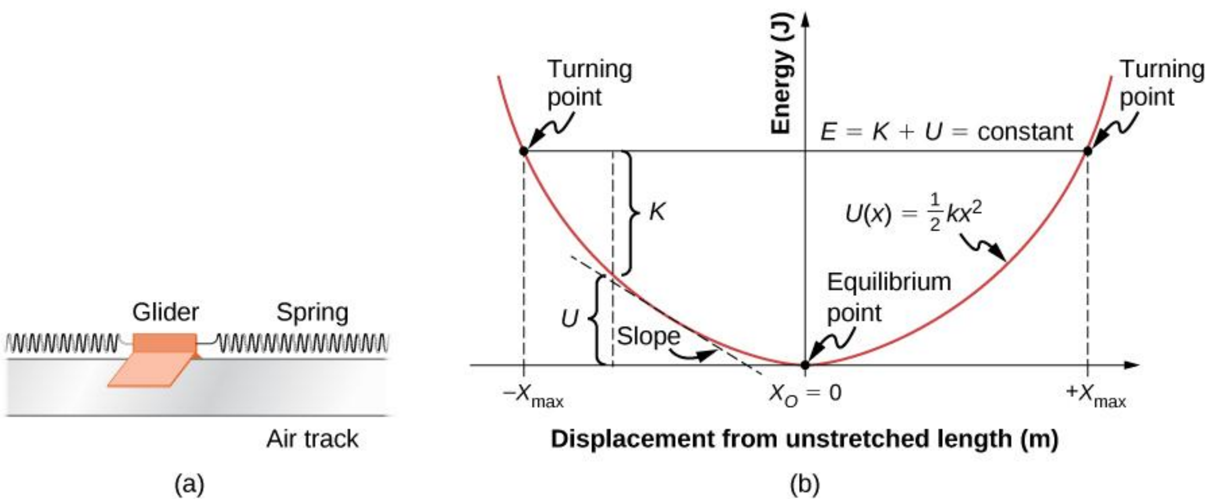
Mechanical Energy (J) : sum of all of a system's kinetic energies and potential energies of all particles

- Mechanical energy is conserved (total change in mechanical energy is 0) if there are only conservative forces present in the system (no friction/drag/resistive forces)
- $E_{mech} = U_g + U_s + KE$

Potential Energy Curves

Potential curves for systems with only conservative forces (gravity, springs, random force)
All forces are generally internal because dependent on displacement so these are true:

- $W = -\Delta U$ or $-W = \Delta U$
- $F(x) = -\frac{dU}{dx}$



Important things to note:

- We don't care about what happens above E_{mech}, it can exist there but doesn't matter to us

How to find:

- **Turning point**: U max, KE min OR value of x where E_{mech} is reached
- **Equilibrium**: U min, KE max OR $\frac{dU}{dt}$ = 0 (guarantees max or min)
  - $\frac{dU^2}{d^2t}$ = + means minimum and solve for x, this will give a stable equilibrium
  - $\frac{dU^2}{d^2t}$ = - means maximum and solve for x, this will give a unstable equilibrium
- **Position**: Find two points on the line of the desired point, then find slope of line, then find y-intercept using y=mx+b with one of the given points, then find the x value with the complete y=mx+b formula

- **Limits of where particle moves**: Find mechanical energy by adding KE and U and solve for the points that reach that mechanical energy
- **Limits of where particle is bound**: minimum potential energy to wherever the endpoint is or where it says "In the limit as r increases without bound, U(r) approaches +x J"
- **Limits of where particle can be found**: limits of x values below the E_{mech}
- **Binding energy of system** or additional energy needed for particle to move to infinity/where it is unbound:
    - E_{mech} + binding energy = energy needed to get to infinity
    - Energy above E_{mech} to get to point where particle goes unbound
- **Speed**: find KE at point, solve for velocity
- **Maximum KE**: x value where it's farthest below E_{mech}
- **KE**: find out how much energy it is below E_{mech} because KE = $|\Delta U|$, KE always positive
- **Force**: $F(x) = -\frac{dU}{dt}$
- **Work**: $\Delta U = -W = -\int_A^B F(r)\,dr$ $\quad W = -\Delta U$
    - $W = \int_A^B F(r)\,dr$
- $\Delta U = U_f - U_o$

Types of Equilibriums

Stable Equilibrium : A small displacement away from the equilibrium results in a restoring force that accelerates a particle back to equilibrium position

- Marble in cup analogy
- Like U shape, concave up
- $F(x) = -\frac{dU}{dt}$

Unstable Equilibrium : A small displacement away from the equilibrium results in a restoring force that accelerates a particle back to equilibrium position

- Rollercoaster analogy
- Like upside down U shape, concave down
- $F(x) = -\frac{dU}{dt}$

Neutral Equilibrium : A small displacement away from the equilibrium results in no restoring force and particle remains in equilibrium without any change in potential energy, continues at rest or in constant velocity

- Flat line
- $F(x) = -\frac{dU}{dt} = 0$

TIPS ON HOW TO SOLVE PROBLEMS:

1. Important formulas:
    a. External agents (applied force): $W = \Delta U$
    b. Internal agents (gravity, spring): $W = -\Delta U$
    c. $W = \int_A^B F(r)\,dr$ ALWAYS
    d. $\Delta U = -W = -\int_A^B F(r)\,dr$ ONLY FOR POTENTIAL CURVES/INTERNAL AGENTS
    e. $F(x) = -\frac{dU}{dt}$ ONLY FOR POTENTIAL CURVES/INTERNAL AGENTS
    f. $\frac{dU}{dt} = 0$ for finding equilibriums (stable, unstable, neutral)
        i. $\frac{dU^2}{d^2t}$ determines which equilibrium type
2. Use $W = F\Delta x$ to find out if work should be positive or negative
3. When looking at equilibriums, try “placing a marble” on the potential energy curve to find the type of equilibrium
4. When looking at a potential energy curve, try “placing a marble” on the potential curve to see how it will move and if it will reach a certain point, will it be moving, etc.
5. $U_g = 0$ is set by you
6. A conservative force can be internal or external, same with non-conservative force

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UNIT 5: System of Particles and Linear Momentum

Fundamentals of Unit 5 Physics

Linear Momentum (kg * m/s) : the quantity that represents the linear progress of motion of a particle or a system

- $p = mv$
- Vector quantity
- Direction is derived from the direction of velocity
- Derivative of momentum with respect to time is equal to net external force
    - $\sum F = ma = m\frac{dv}{dt}$

- $\sum F = \frac{d}{dt}(mv)$ Mass can also change
- $\sum F = \frac{dp}{dt}$

- Conservation of momentum = momentum remains constant = no net external force
  - $\Delta p = 0$ means $\sum F_{external} = 0$

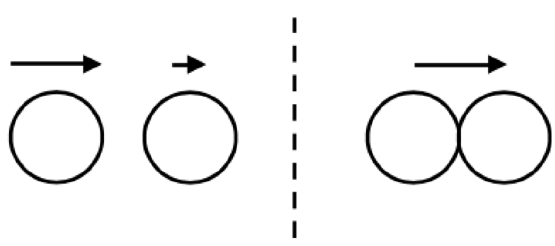
Impulse (kg * m/s or N * s) : change in momentum of a particle or a system

- Vector quantity
- Direction is derived from direction of force
- Area under curve of a force vs time graph
- Deriving
  - $\sum F_{ext} = \frac{dp}{dt}$
  - $dp = dt \sum F_{ext}$
  - $\int dp = \int dt \sum F_{ext}$
  - $p_{f,} - p_o = \sum F_{ext} \Delta t$
  - $J = \Delta p = \sum F_{ext} \Delta t$ (constant force and time independent)

  - $\sum F_{ext} = \frac{dp}{dt}$
  - $dp = \sum F_{ext}\, dt$
  - $\int dp = \int \sum F_{ext}\, dt$
  - $J = \Delta p = \int \sum F_{ext}\, dt$ (variable force and time dependent) or (integrate F(t))
  - $J = \int F\, dt$
- Average force
  - Get impulse first (with change in momentum)
  - Then divide by Δt
  - $\Delta p = F_{avg} \Delta t$
  - $F_{avg} = \Delta p / \Delta t$
  - Greater impact time means lower force

Types of Collisions

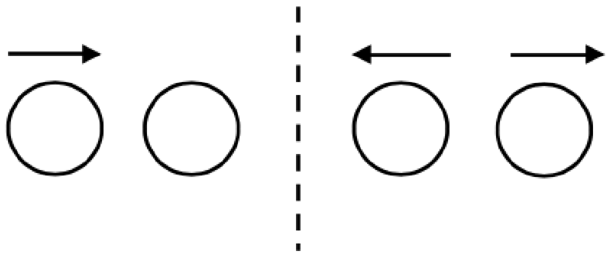
Inelastic Collision :



- Momentum conserved and Kinetic energy is lost
- Occurs when two objects collide and stick together

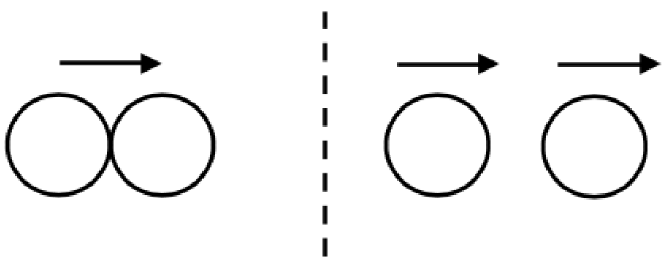
- $\Delta p = 0$ $\sum F_{ext} = 0$ $\Delta KE < 0$
- $m_1v_1 + m_2v_2 = (m_1 + m_2)v'$

Elastic Collision :



- Momentum and Kinetic energy is conserved
- Occurs when two objects collide and bounce off each other
- Ex: bouncy ball
- $\Delta p = 0$ $\sum F_{ext} = 0$ $\Delta KE = 0$
- $m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$
- $v_1' = (\frac{m_1 - m_2}{m_1 + m_2})v_1 + (\frac{2m_1}{m_1 + m_2})v_2$
- $v_2' = (\frac{2m_1}{m_1 + m_2})v_1 + (\frac{m_2 - m_1}{m_1 + m_2})v_2$
- Can solve for final velocities also using Center of Mass velocity (V_{cm}) [Look at Center of Mass]
    - $v_1' = 2v_{cm} - v_1$
    - $v_2' = 2v_{cm} - v_2$

Superelastic (Explosion) Collision :



- Momentum is conserved and Kinetic energy increases
- Occurs when two objects are together and come apart
- Ex: astronaut pushing off of spaceship, rocket
- $\Delta p = 0$ $\sum F_{ext} = 0$ $\Delta KE > 0$
- $(m_1 + m_2)v = m_1v_1' + m_2v_2'$

Center of Mass

Center of Mass : point in space where the mass of all particles of the system can be concentrated
- Behaves like an actual mass and the behavior is very predictable
- Take time derivatives to go from $x_{cm} \rightarrow v_{cm} \rightarrow a_{cm}$

- $x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{\sum m_ix_i}{m_1 + m_2} = \frac{1}{M}\sum m_ix_i$ M = total mass of system
- $v_{cm} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} = \frac{\sum m_iv_i}{m_1 + m_2} = \frac{1}{M}\sum m_iv_i$ $\sum m_iv_i$ = total momentum in system
- $a_{cm} = \frac{m_1a_1 + m_2a_2}{m_1 + m_2} = \frac{\sum m_ia_i}{m_1 + m_2} = \frac{1}{M}\sum m_ia_i$ $\sum m_ia_i$ = net external force
- COM in multiple dimensions
    - $r_{cm} = x_{cm}i + y_{cm}j + z_{cm}k = \frac{1}{M}\sum m_ix_i = \frac{1}{M}\int x\,dm$
- Using Center of Mass for solving final velocities for inelastic collision
    - Draw diagram
    - Calculate v_{cm}
    - Subtract v_{cm} from both velocities (v_1 and v_2) to get U_1 and U_2
    - Flip signs to get U_1' and U_2'
    - Add v_{cm} to U_1' and U_2' to get final velocities (v_1' and v_2')
    - $v_1' = 2v_{cm} - v_1$
    - $v_2' = 2v_{cm} - v_2$

Linear Mass Density : constant through a uniform object, mass per unit of length is constant
- There's also area mass density and volume mass density
- $\frac{M}{L} = \frac{dm}{dx}$
- $\frac{M}{L}dx = dm$
- $\frac{1}{M}\int x\,dm \rightarrow \frac{1}{M}\int x\frac{M}{L}dx \rightarrow \frac{1}{L}\int xdx \rightarrow \frac{L^2}{2L} - 0 \rightarrow \frac{L}{2}$

2D Momentum

Momentum in a 2D plane
- Momentum in each direction is conserved
- $P_o = P_f$
- $p_xi + p_yj = p_xi + p_yj$
- $P_f = mv_f = p_xi + p_yj$
- $v_f = (p_xi + p_yj) / m_1 + m_2$
- $\theta = \tan^{-1}(p_y / p_x) = \tan^{-1}(v_y / v_x)$

Example: Cue ball and a pool ball being hit off center
If $m_1 = m_2$ and $v_2 = 0$, the angle between m_1 and m_2 after their collision is ALWAYS 90˚ (pool ball scenario)

When doing 2D momentum problems, treat it as a normal momentum problem but split it into each direction

Exploding Projectiles

Rocket exploding at top of the trajectory (apex), how far will the warhead go?

Using kinematics:

- Find speed, time, and displacement before the collision (explosion)
- Use conservation of momentum to find speed after collision
- Find time and displacement after the collision (explosion)
- Add displacement before and after collision

Using COM: (easier)

- Find the time to the apex and multiply by 2 to get time until COM hits the ground again
- Use the total time to find the range of COM
- Use COM formula ($x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$) to get x_2 $x_{cm} = R$, $x_1 = R/2$

Ballistics Pendulum

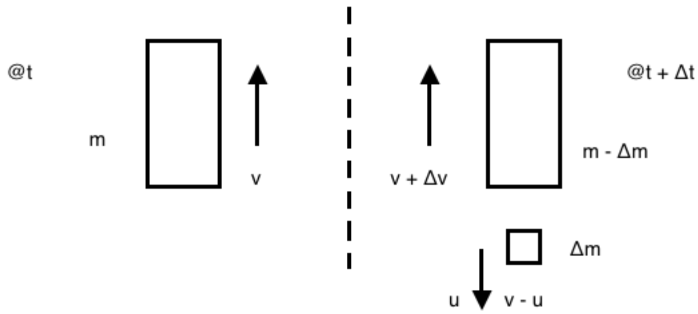
Shooting a bullet at a block attached to a string, find the initial speed of the bullet.

Use conservation of momentum to find expression for the initial speed of the bullet
Use conservation of mechanical energy to find expression for the final speed of the block and bullet after collision

$V = (\frac{m+M}{m})\sqrt{2gL(1 - cos\Theta)}$ m/s

Rocket Propulsion

Mass and speed of a rocket changes, we need to account for this change.



Conservation of momentum is applied to find the initial and final momentum of all objects

- u = speed of exhaust gas
- dm/dt = rate of rocket's mass change with respect to time

Rocket Equation: $0 = ma - u\frac{du}{dt}$

- $u\frac{du}{dt}$ = thrust
- ma = force

Burn in Space Equation (without gravity's pull): $v_f - v_o = -uln(\frac{m_f}{m_o})$

Launch from Earth Equation: $v_f - v_o = -uln(\frac{m_f}{m_o}) - gt$

- gt is the "price" or loss in final speed as a result of launching from Earth

TIPS ON HOW TO SOLVE PROBLEMS:

1. Important formulas:
    a. $\sum F = \frac{dp}{dt}$ (net external force = derivative of momentum)
    b. $J = \int F\, dt$ (impulse = integral of force) (reverse of equation a)
    c. $J = \Delta p$ (impulse = change in momentum)
    d. $J = \sum F_{ext}\, \Delta t$ (impulse = net external force * change in time/impact time)
    e. $J = F_{avg}\, \Delta t$ (impulse = average force * change in time/impact time)
    f. $\Delta p = 0$ $\sum F_{ext} = 0$ (conservation of momentum and net external force = 0)
    g. $v_1' = 2v_{cm} - v_1$ AND $v_2' = 2v_{cm} - v_2$ (equations using COM to find final speed of elastic collision)
    h. $x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{1}{M}\sum m_i x_i$ (position of COM)
    i. $v_{cm} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} = \frac{1}{M}\sum p_i$ (velocity of COM, relates to total momentum)

j. $a_{cm} = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2} = \frac{1}{M}\sum F_{ext}$ (acceleration of COM, relates to net external force)

k. $\frac{M}{L} = \frac{dm}{dx}$ (linear density)

l. $v_f = (p_x i + p_y j) / m_1 + m_2$ (finding speed from 2D momentum)

m. $\theta = \tan^{-1}(p_y / p_x) = \tan^{-1}(v_y / v_x)$ (finding angle from 2D momentum)

n. $V = (\frac{m+M}{m})\sqrt{2gL(1 - cos\Theta)}$ m/s (initial speed of bullet for ballistics pendulum)

o. $0 = ma - u\frac{du}{dt}$ (rocket equation)

p. $v_f - v_o = -uln(\frac{m_f}{m_o})$ (burn in space equation)

q. $v_f - v_o = -uln(\frac{m_f}{m_o}) - gt$ (launch from Earth equation)

2. Find impact time by setting the unit divided by time
   a. 100 bullets/min → 60 seconds/100 bullets = 0.6 sec/bullet
   b. Impact type is a “cycle” of having force applied plus not being applied, this cycle starts again when force is applied again
   c. Use this impact time to find F_{avg} using $J = F_{avg} * \Delta t$
3. Use Center of Mass to find the final speeds of elastic collisions and displacement of an object when exploding in midair (exploding projectiles/missiles)

UNIT 6: Rotational Motion

Fundamentals of Unit 6 Physics

Rotational Kinematics : describing rotational motion and how it's occurring
- Angular position = θ
- Angular velocity = $\omega = d\theta/dt$
- Angular acceleration = $\alpha = d\omega/dt = d^2\theta/dt^2$

Rotational Dynamics : why is the rotational motion occurring
- Torque, angular collisions, angular momentum

Right Hand Rule : used to determine the direction of angular quantities
- Angular velocity, acceleration, momentum, torque, etc.
- "Sweep" position vector into force = torque
- "Sweep" position vector in velocity = angular momentum

Torque : the ability of a force to rotate an object about an arbitrary axis of own choice
- $\tau = I\alpha$ [N2LR]
- $\tau = r_\perp \times F$
- $|\tau| = |r|\,|F| \sin\theta$
  - θ = angle between r and F vectors
  - r = position vector measured from axis of rotation
  - F = force

Linear vs Angular Table

Linear Name	Linear	Transition Equation	Angular	Angular Name
Position	x	$x = r\theta$	θ	Angular position
Velocity	v	$v = r\omega$	ω	Angular velocity
Acceleration	a	$a = r\alpha$	α	Angular acceleration
Mass	m	$I = fmr^2$	I	Moment of Inertia
Kinetic Energy	$KE = ½mv^2$	$KE_r = ½(fmr^2)(v/r)^2$	$KE_r = ½I\omega^2$	Kinetic Energy Rotational
Force	$F = ma$ $= dp/dt$	$\tau = (mr^2)(a/r)$ $= r * ma$ $= r\sin\theta * F$ $= r_\perp \times F$	$\tau = I\alpha$ $= r_\perp \times F$ $= dL/dt$	Torque
Work	$W = F\Delta x$		$W = \tau\Delta\theta$	Work

Power	$P = dW/dt$ $= Fv$ (force constant)		$P = dW/dt$ $= \tau\omega$ (torque constant)	Power
Momentum	$p = mv$	$L = r \times p$	$L = I\omega$ $= r_{\perp} \times p$ $= r_{\perp}mv$	Angular Momentum
Impulse	$J = \Delta p$		$? = \Delta L$	Angular Impulse
Work - KE theorem	$W = \Delta KE$ $F\Delta x = \Delta KE$		$W = \Delta KE_r$ $\tau\Delta\theta = \Delta KE_r$	Work - KE rotational theorem
Impulse - Momentum Theorem	$F\Delta t = \Delta p$ $F\Delta t = m\Delta v$		$\tau\Delta t = \Delta L$ $\tau\Delta t = I\Delta\omega$	Angular Impulse - Momentum Theorem

Angular Kinematics

1. $V_f = V_o + at$ $\qquad\qquad$ $\omega_f = \omega_o + \alpha t$
2. $\Delta x = V_o t + ½at^2$ $\qquad\qquad$ $\Delta\theta = \omega_o t + ½\alpha t^2$
3. $V_f^2 = V_o^2 + 2a\Delta x$ $\qquad\qquad$ $\omega_f^2 = \omega_o^2 + 2\alpha\Delta\theta$

Moment of Inertia

Moment of Inertia (kg * m^2) : quantitative measure of difficulting in rotating a rigid object or a collection of masses about an arbitrary axis

- Bigger = harder to move
- Smaller = easier to move
- $I = fmr^2$
- f = shape factor
    - Inertias for common objects:
    - **Ring**/Hoop/Thin Cylindrical Shell = MR^2 $\qquad$ f = 1
    - **Cylinder**/Solid Cylinder/Disk = $½MR^2$ $\qquad$ f = ½
    - Solid **Sphere** = $⅖MR^2$ $\qquad$ f = ⅖
    - **Shell**/Thin Spherical Shell = $⅔MR^2$ $\qquad$ f = ⅔

Parallel Axis Theorem

An easy way to get the new inertia of an object when its axis of rotation changes

$I_{new} = I_{old} + mD^2$

I_{old} = old inertia

m = mass of object

D = distance between old and new axis of rotation

Rotational Kinetic Energy

$KE_r = ½I\omega^2$

$KE_r = ½(fmr^2)(v/r)^2$

$KE_r = ½fmv^2$

- There is KE_r present if an object is rolling, not slipping
- Essentially linear KE but factors in shape factor

Newton's Laws of Rotational Motion

N1L : $\Sigma F = 0$
N1LR : $\Sigma\tau = 0$

N2LR : $\Sigma F = ma$
N2LR : $\Sigma\tau = I\alpha$

Rolling friction

- Generally very small $\mu_r = F_{fr}$
- Causes objects to roll
- Still affects acceleration as an object rolls (up or down an incline)

Energy consideration in rotational motion

Work - Kinetic energy theorem
- $W = \Delta KE_r$
- $\tau\Delta\theta = \Delta KE_r$

Power from Torque
- $P = dW/dt$
- $P = \tau\omega$

Conservation of Mechanical Energy in rotational motion
- $\Delta E_{mech} = 0$
- $U_{go} + K_o + K_{Ro} = U_{gf} + K_f + K_{Rf}$
- $mgh + \frac{1}{2}mv_o^2 + \frac{1}{2}I\omega_o^2 = mgh + \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2$

Rolling motion of a rigid object

Pure rotation
- $v' = r\omega$ [Speed of edge of wheel]
- $v_{cm} = 0$ [Speed of center of wheel]

Pure translation
- $v' = v_{cm}$ [Speed of edge of wheel is the same as center of wheel]

Pure rolling
- $v_{cm} = ds/dt = r * d\theta/dt = r\omega$
- $v' = 2v_{cm} = 2r\omega$ [Speed of edge of wheel is twice as fast as the center of wheel]

Angular Momentum

- $L = r \times p = rmv = I\omega$
- $\Delta L = \tau * t$

- Torque changes angular momentum

Conservation of Angular Momentum

- $\Delta L = 0$ $\Sigma\tau_{external} = dL/dt = 0$
- $L_o = L_f$
- L can be Rmv or Iω

Angular Impulse - Momentum Theorem

- $\tau\Delta t = \Delta L$
- $\tau\Delta t = I\Delta\omega$

TIPS ON HOW TO SOLVE PROBLEMS:

1. When doing angular momentum problems, use $r_{\perp}mv$ for objects that aren't rotating but have angular momentum around a point (ball going towards bar) and use Iω for objects that are rotating around a point (bar itself)
2. If axis of rotation changes after a collision, write initial quantities in terms of the new axis of rotation to keep the axis of rotation constant, rather than changing it before and after collision
3. To understand the direction of rotational friction, understand how angular velocity is changing, it doesn't always oppose translational motion (friction points up an incline as ball is rolling up)
4. To get acceleration of an object up or down an incline, it's easier to use parallel axis theorem and have the gravity acting on COM to be applying torque rather than have rolling friction applying torque and solving for acceleration

UNIT 7: Oscillations

Fundamentals of Unit 7 Physics

Angular Frequency (ω) [rad/s] : ω is constant for a specific system, not the same as frequency or angular velocity

- $\omega = 2\pi f$

Phase constant (φ) [rad] : phase angle or "phaser", tells us when we started counting time and where the mass was when stopwatch started, no physics is involved with this constant

Amplitude (A) [meters] : location of mass when $t = 0$ and $A = X_{max}$ (why cosine function is better solution for differential equations)

Frequency (f) [Hz] : oscillations or cycles per second

- $f = 1/T$

Period (T) [seconds] : time it takes for 1 cycle to be completed

- $T = 1/f$

Oscillations require solving differential 2nd order equations in order to get x(t)

<table>
<tr><th>General</th><th>Mass-Spring System</th><th>Simple Pendulum System</th></tr>
<tr>
<td>$\omega = 2\pi f = 2\pi/T$
$\omega = \sqrt{\frac{dmg}{I_P}}$ [rigid pendulum object]

$f = 1/T$
$f = \frac{1}{2\pi}\sqrt{\frac{dmg}{I_P}}$ [rigid pendulum object]

$T = 1/f$
$T = 2\pi\sqrt{\frac{I_P}{dmg}}$ [rigid pendulum object]

d = distance between pivot/axis of rotation and COM</td>
<td>Restoring Force: $-F_s = -k\Delta x$
$\omega = \sqrt{\frac{k}{m}}$

$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$
$T = 2\pi\sqrt{\frac{m}{k}}$

$x(t) = X_{max}\cos(\omega t + \boldsymbol{\varphi})$
$v(t) = -X_{max}\omega\sin(\omega t + \boldsymbol{\varphi})$
$v(t) = -X_{max}\omega^2\cos(\omega t + \boldsymbol{\varphi})$

Total Energy = $E_{mech} = ½kX_{max}^2 = ½mV_{max}^2$

$v(x) = \omega\sqrt{A^2 - x^2}$
$V_{max} = \omega A = \omega X_{max} = \sqrt{\frac{k}{m}}X_{max}$

$a_{max} = \omega V_{max}$</td>
<td>Restoring Force: $-mg\sin\theta$
$\omega = \sqrt{\frac{g}{L}}$

$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$
$T = 2\pi\sqrt{\frac{L}{g}}$

$\theta <= 5°$ (small angle approx $\sin\theta = \theta$)
$x(t) = \theta_{max}\cos(\omega t + \boldsymbol{\varphi})$
$v(t) = -\theta_{max}\omega\sin(\omega t + \boldsymbol{\varphi})$
$v(t) = -\theta_{max}\omega^2\cos(\omega t + \boldsymbol{\varphi})$</td>
</tr>
<tr><td colspan="3">Common periods for different pendulum objects:
Uniform rod pivoting @ an end: $T = 2\pi\sqrt{\frac{2L}{3g}}$
Uniform loop pivoting @ circumference: $T = 2\pi\sqrt{\frac{3r}{2g}}$
Uniform ring pivoting @ circumference: $T = 2\pi\sqrt{\frac{2r}{g}}$</td></tr>
</table>

Damped Oscillations

Damping Coefficient (b) : constant

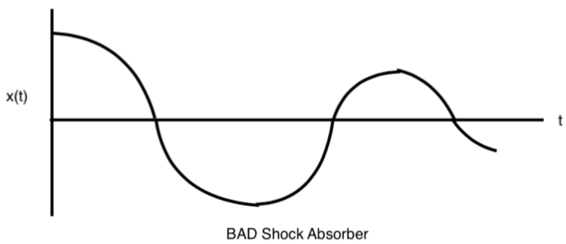
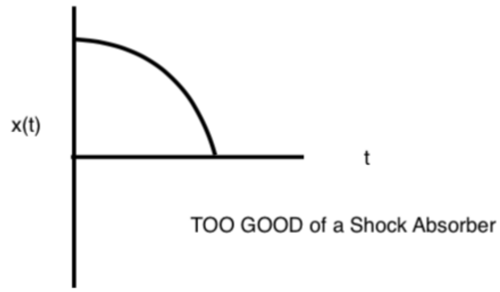
- Buoyancy force always opposes motion

Damped Oscillations : oscillations that slowly dissipate until it stops oscillating

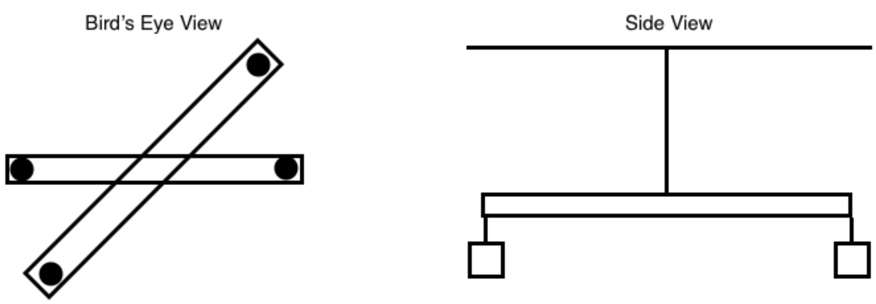
- Equation of Motion
  - $x(t) = Ae^{-(\frac{b}{2m})t}\cos(\omega t + \boldsymbol{\varphi})$
  - $\omega = \sqrt{\omega_o^2 + (\frac{b}{2m})^2}$

  - b = damping coefficient
  - m = mass of object oscillating
  - ω = angular frequency

- ω_o = natural frequency of system = $\sqrt{\frac{k}{m}}$

Underdamped System	Overdamped (Critically Damped) System
$\omega_o = \sqrt{\frac{k}{m}}$ $\omega_o > \sqrt{\frac{b}{2m}}$  	$\omega_o = \sqrt{\frac{k}{m}}$ $\omega_o < \sqrt{\frac{b}{2m}}$ 

Torsional Pendulum



Setup - twisting the pendulum, releasing it, and measuring the period (change in direction to change in direction)

$\theta(t) = \theta_{max}\cos(\omega t + \boldsymbol{\varphi})$

$\omega = \sqrt{\frac{K}{I}}$

- K = kappa = torsional constant of the wire
    - Depends on material, length, cross-sectional diameter

$f = \frac{1}{2\pi}\sqrt{\frac{K}{I}}$

$T = 2\pi\sqrt{\frac{I}{K}}$

Forced Oscillation and Resonance

Imposed Frequency : a frequency imposed onto a system

- Moving a mass on a spring up and down then letting go to allow it to oscillate by itself
- $F_o\cos(\omega t)$ = periodic frequency forced upon system

DON'T CONFUSE OMEGAS

ω = imposed frequency

ω_o = natural frequency = $\sqrt{\frac{k}{m}}$

A = displacement from equilibrium

F_o = imposed force

m = mass of object

Equation of Motion:

$$A(\omega) = \frac{F_o / m}{(\frac{k}{m} - \omega^2)}$$

$$A(\omega) = \frac{F_o / m}{(\omega_o^2 - \omega^2)}$$

Case 1: imposed frequency is much smaller than the natural frequency

If $\omega << \omega_o$ then $\omega_o^2 - \omega^2 = \omega_o^2$

$$A(\omega) = \frac{F_o / m}{\omega_o^2} = \frac{F_o}{m} * \frac{m}{k} = \frac{F_o}{k}$$

A approaches $\frac{F_o}{k}$

Case 2: imposed frequency is much larger than the natural frequency

If $\omega >> \omega_o$ then $\omega \to \infty$

$$A(\omega) = \frac{F_o / m}{(\omega_o^2 - \omega^2)} \text{ as } \omega \to \infty$$

$A(\omega) \to 0$

A approaches 0

Case 3: imposed frequency equals natural frequency and the resonance phenomenon is reached

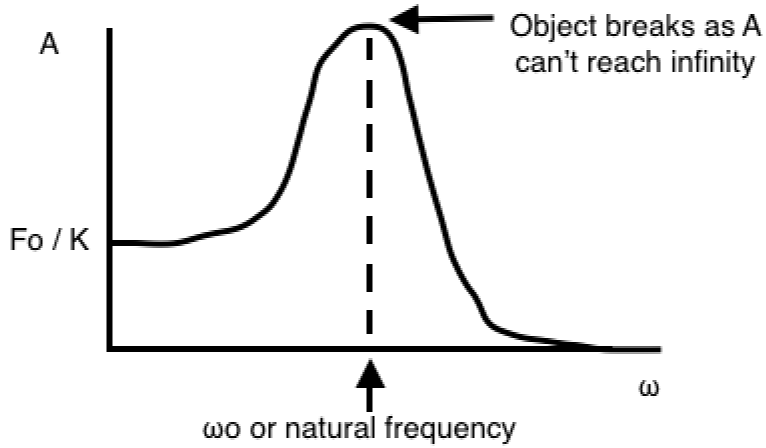
If $\omega == \omega_o$

$$A(\omega) = \frac{F_o / m}{(\omega_o^2 - \omega^2)} = \frac{F_o / m}{0}$$

$A(\omega) \to \infty$

A approaches ∞ but it can't, so the object breaks

- Ex: breaking wine glass with voice using resonant frequency



TIPS ON HOW TO SOLVE PROBLEMS:

1. List out all your knowns and try to find any unknown using the general equations
    a. $\omega = 2\pi f$, $f = 1/T$, $T = 1/f$
2. Understand the difference between angular frequency, natural frequency, and imposed frequency
3. Might be helpful to memorize angular frequencies for common objects and scenarios

UNIT 8: Gravitation

Fundamentals of Unit 8 Physics

Newton's Gravitational Law :

- $F_g = \frac{Gm_1m_2}{r^2}$
- $G = 6.67 \times 10^{-11}\ N * m^2 / kg^2$
- r = distance between m_1 and m_2
- Only applied to point masses
- All masses apply a gravitational force on all other masses

Gravitational Potential Energy

$U_g(r) = \frac{-Gm_1m_2}{r}$

[negative indicates that energy is REQUIRED to place mass m_1 at "r" distance away from mass m_2]

Energy of Assembled Masses

- It takes no work to place a mass in an empty space (no other masses present)
- The next mass placed will take $\frac{-Gm_1m_2}{r}$ amount of work to place the second mass
- The next mass placed will take $\frac{-Gm_1m_2}{r}$ amount of work from the first and second mass to place the third mass
- The sum of all works is Total Binding Energy of the System

Mechanical Energy of a Star Planet System

$E_{mech} = U_g + K = \frac{-Gm_1m_2}{r} + \frac{1}{2}mv^2$

$F_g = F_c$	[Force of gravity is centripetal force holding planet in orbit]
$\frac{Gm_1m_2}{r^2} = \frac{m_2v^2}{r}$	[m_2 mass of planet]
$\frac{-Gm_1m_2}{r} = m_2v^2$	[simplify to mv^2]
$E_{mech} = \frac{-Gm_1m_2}{r} + \frac{Gm_1m_2}{2r}$	[Substitute into E_{mech} for mv^2]
$E_{mech} = \frac{-Gm_1m_2}{2r}$	[Binding energy of a star-planet system, energy required to free planet from orbit] [r = radius or semi-major axis if orbit isn't elliptical]

Escape Velocity

- Using Conservation of Mechanical Energy, we can find velocity needed to escape Earth's gravity
- Assume:
    - $U_g = 0$ at infinity distance away from Earth
    - The object ends with 0 velocity in space

$\Delta E_{mech} = 0$

$U_{go} + K_o = U_{gf} + K$

$\frac{-Gm_1m_2}{r} + ½mv_{esc}^2 = 0 + 0$

$½m_2v_{esc}^2 = \frac{Gm_1m_2}{r}$

$v_{esc} = \sqrt{\frac{2Gm_1}{r}}$ [m_1 = mass of planet, r = radius of planet]

V_{esc} for Earth is 11.2 km/s, spaceships need to exceed this at one point in order to enter space

Kepler's Laws of Planetary Motion

Kepler's 1st Law : All planets orbit their star in an elliptical path, with the star located at one of the foci of the elliptical orbit

Kepler's 2nd Law : an imaginary line drawn from the center of the star to the center of its planet sweets equal area in space in equal time intervals

- dA/dt = L/2m [rate of change = constant value]

Kepler's 3rd Law : the square of a planet's period around it star is directly proportional to the cubed of the semi-major axis of the planet's orbit

- $T^2 \propto a^3$ [elliptical orbit]
- $T^2 \propto r^3$ [circular orbit]
- $F_g = F_c$ and $v = 2\pi r / T$
- $(\frac{4\pi^2}{GM_s})\, r^3 = T^2$ [M_s = mass of sun, $(\frac{4\pi^2}{GM_s})$ = kepler's constant]

Spy/weather Satellites and Communication/Geosynchronous Satellites

$(\frac{4\pi^2}{GM_s})\, r^3 = T^2$

- "r" can be found
- Altitude can then be found
- "V" or speed can be found with $v = 2\pi r / T$

A satellite is geosynchronous if its period is 24 hours long, stays in same place relative to Earth's surface

TIPS ON HOW TO SOLVE PROBLEMS:

1. For objects far from planet surface, set Ug = 0 at center of the planet or at infinity distance away, who knows why we can't just set Ug = 0 at the planet's surface